

PIERI TYPE RULES AND $Gl(2|2)$ TENSOR PRODUCTS

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ABSTRACT. We derive a closed formula for the tensor product of a family of mixed tensors using Deligne's interpolating category $Rep(Gl_0)$. We use this formula to compute the tensor product between any two maximal atypical irreducible $Gl(2|2)$ -representations.

INTRODUCTION

For the classical group $Gl(n)$ the tensor product decomposition

$$L(\lambda) \otimes L(\mu) = \bigoplus_{\nu} c_{\lambda\mu}^{\nu} L(\nu)$$

between two irreducible representations is given by the Littlewood-Richardson rule. Contrary to this case the analogous decomposition between two irreducible representation of the General Linear Supergroup $Gl(m|n)$ is poorly understood. A classical result from Berele and Regev [BR87] and Sergeev [Ser85] shows that the fusion rule between direct summands of tensor powers $V^{\otimes r}$ of the standard representation $V \simeq k^{m|n}$ is again given by the Littlewood-Richardson rule. The first more general results were achieved in [Hei14] where we obtained a decomposition law between any two mixed tensors, direct summands in the mixed tensor space $V^{\otimes r} \otimes (V^{\vee})^{\otimes s}$, $r, s \in \mathbb{N}$. This result is based on the tensor product decomposition in Deligne's interpolating category $Rep(Gl_{\delta})$ [Del07]. This category comes for $\delta = m - n$ with a tensor functor $F_{m|n} : Rep(Gl_{m-n}) \rightarrow Rep(Gl(m|n))$. Since the decomposition for the tensor product of two indecomposable elements is known for $Rep(Gl_{m-n})$ by results from Comes and Wilson [CW11], we obtain an analogous law once we describe the image of $F_{m|n}(X)$ for indecomposable objects X in $Rep(Gl_{m-n})$. This in turn is based on results by Brundan and Stroppel [BS12b] on the interplay between Khovanov algebras and Walled Brauer algebras. Since any Kostant module and any projective representation is a mixed tensor (up to some Berezin twist), these results give a decomposition law for their tensor products.

These irreducible representations are still special. For instance no nontrivial maximal atypical irreducible representation of $Gl(n|n)$ is a mixed tensor. In this article we derive a closed formula for the tensor product $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ of a family of mixed tensors in $Rep(Gl(n|n))$ and use this formula to compute the tensor product of any two maximal atypical $Gl(2|2)$ -representations. We hope that these results shed some light on this very difficult problem. In the $Gl(2|2)$ -case the irreducible representations are either typical, singly atypical or double (maximal) atypical. Every typical representation is a mixed tensor and every singly atypical irreducible representation is a Berezin twist of a mixed tensor. Hence the results of [Hei14] give the decomposition law for tensor products between typical and/or singly atypical

irreducible representations. In [Hei14] we also explain how to decompose the tensor products between a typical and an irreducible maximal atypical representation in the $Gl(2|2)$ -case. Hence the fusion laws between irreducible representations are known except for i) the tensor product of a singly atypical and a maximal atypical representation and ii) the tensor product between two maximal atypical representations. Here we focus on the 2nd case and obtain in corollary 6.6 a formula for their tensor products. Similar, computer-based formulas have been obtained before in the more restrictive $\mathfrak{psl}(2|2)$ -case [GQS05]. The maximal atypical representations are particularly interesting. For instance an irreducible representation has nonvanishing superdimension if and only if it is maximally atypical [Ser10] [Wei10] and every block of atypicality k is equivalent to the unique maximal atypical block of $Gl(k|k)$ [Ser06]. In particular the fusion rules in the maximal atypical case are essential in our study of the quotient category $Rep(Gl(n|n))/\mathcal{N}$ in [HWng]. Every irreducible maximal atypical representation for $Gl(2|2)$ is of the form $[a, b] = L(a, b) - b, -a$, and any such representation is a Berezin twist of one of the $S^i := [i, 0]$. More generally we also consider the representations $S^i := [i, 0, \dots, 0]$ in $Rep(Gl(n|n))$. Any such representation can be realised as the unique constituent of highest weight in a mixed tensor \mathbb{A}_{S^i} . The socle filtration of the \mathbb{A}_{S^i} is known [Hei14]. We split the computation of $S^i \otimes S^j$ into two parts. We first project onto the maximal atypical block Γ and then compute the remaining summands afterwards. We derive a closed formula for the projection of $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ onto Γ in section 3 and consider the resulting equality in the Grothendieck ring K_0 . $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ splits into representations of the form $\mathbb{A}_{S^{\dots}}$ and mixed tensors $R(a, b)$ whose composition factors are known in the $Gl(2|2)$ -case. In $[\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}] \in K_0$ the tensor product $S^i \otimes S^j$ occurs exactly once, and all other tensor products are of the form $Ber^{\dots}(S^k \otimes S^l)$ with either k, l smaller than i, j . This allows us to compute the maximal atypical composition factors of $S^i \otimes S^j$ recursively in lemma 5.2. In order to determine the decomposition into maximal atypical indecomposable representations we use the theory of cohomological tensor functors [HW14]. Here we consider the tensor functor $DS : Rep(Gl(2|2)) \rightarrow Rep(Gl(1|1))$. The main theorem of [HW14] gives a formula for $DS(L)$ for any irreducible representation and we get $DS(S^i) = Ber^i \oplus \Pi Ber^i$ where Π denotes the parity shift. This gives us strict estimates on the number of indecomposable summands and their superdimension which is enough to get the final result 6.6. In section 6 we compute the indecomposable summands which are not maximal atypical. The remaining composition factors in $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ are all $(n-2)$ -atypical, hence projective for $n=2$. Hence they cannot combine to an indecomposable representation and the K_0 -decomposition is enough for the computation. These methods allow in principle to compute the decomposition $S^i \otimes S^j$ for any n . However it is very difficult to determine the composition factors of the mixed tensors $R(a, b)$ for $n \geq 3$. We end the article with a conjecture concerning the decomposition of $S^i \otimes S^j$ and its socle for arbitrary n .

1. THE SUPERLINEAR GROUPS

Let k be an algebraically closed field of characteristic zero. Let $\mathfrak{g} = \mathfrak{gl}(m|n) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the general linear superalgebra and $Gl(m|n)$ the general linear supergroup. By definition a finite dimensional super representation ρ of $\mathfrak{gl}(m|n)$ defines a representation ρ of $Gl(m|n)$ if its restriction to \mathfrak{g}_0 comes from an algebraic representation of $G_{\overline{0}}$, also denoted ρ . We denote the category of finite-dimensional

representations with parity-preserving morphisms by $T = T_{m|n}$. For $M \in T$ we denote by M^\vee the ordinary dual and by M^* the twisted dual. For simple and for projective objects M of T we have $M^* \cong M$.

The category \mathcal{R} . Fix the morphism $\varepsilon : \mathbb{Z}/2\mathbb{Z} \rightarrow G_{\overline{0}} = Gl(n) \times Gl(n)$ which maps -1 to the element $diag(E_n, -E_n) \in Gl(n) \times Gl(n)$ denoted ϵ_{nn} . We write $\epsilon_n = \epsilon_{nn}$. Notice that $Ad(\epsilon_{nn})$ induces the parity morphism on the Lie superalgebra $\mathfrak{gl}(n|n)$ of G . We define the abelian subcategory \mathcal{R} of T as the full subcategory of all objects (V, ρ) in T with the property $p_V = \rho(\epsilon_{nn})$; here ρ denotes the underlying homomorphism $\rho : Gl(n) \times Gl(n) \rightarrow Gl(V)$ of algebraic groups over Λ . The subcategory \mathcal{R} is stable under the dualities $^\vee$ and * . For $G = Gl(n|n)$ we usually write T_n instead of T , and \mathcal{R}_n instead of \mathcal{R} , to indicate the dependency on n . The irreducible representations are indexed by weights with respect to the standard Borel subalgebra of upper triangular matrices. We denote by $L(\lambda)$ the irreducible representation with highest weight $\lambda = (\lambda_1, \dots, \lambda_n | \lambda_{n+1}, \dots, \lambda_{n+n})$. The Berezin determinant of $Gl(n|n)$ defines a one dimensional representation $B = Ber$ with weight $(1, \dots, 1 | -1, \dots, -1)$. An object $M \in T_n$ is called negligible, if it is the direct sum of indecomposable objects M_i in T_n with superdimensions $sdim(M_i) = 0$. The ideal of negligible objects is denoted \mathcal{N} or \mathcal{N}_n .

2. MIXED TENSORS

Let MT denote the full subcategory of mixed tensors in \mathcal{R}_n whose objects are direct sums of the indecomposable objects in \mathcal{R}_n that appear in a decomposition $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$ for some natural numbers $r, s \geq 0$, where $V \in \mathcal{R}_n$ denotes the standard representation. By [BS12b] and [CW11] the indecomposable objects in MT are parametrized by $(n|n)$ -cross bipartitions. Let $R_n(\lambda)$ (or $R(\lambda)$ if the dependency on n is clear) denote the indecomposable representation in \mathcal{R}_n corresponding to the bipartition $\lambda = (\lambda^L, \lambda^R)$ under this parametrization. To any bipartition we attach a weight diagram in the sense of [BS11], ie. a labelling of the numberline \mathbb{Z} according to the following dictionary. Put

$$I_\wedge(\lambda) := \{\lambda_1^L, \lambda_2^L - 1, \lambda_3^L - 2, \dots\} \quad \text{and} \quad I_\vee(\lambda) := \{1 - \lambda_1^R, 2 - \lambda_2^R, \dots\}.$$

Now label the integer vertices i on the numberline by the symbols $\wedge, \vee, \circ, \times$ according to the rule

$$\begin{cases} \circ & \text{if } i \notin I_\wedge \cup I_\vee, \\ \wedge & \text{if } i \in I_\wedge, i \notin I_\vee, \\ \vee & \text{if } i \in I_\vee, i \notin I_\wedge, \\ \times & \text{if } i \in I_\wedge \cap I_\vee. \end{cases}$$

To any such data one attaches a cup-diagram as in section [CW11] or [BS11] and we define the following three invariants

$$\begin{aligned} rk(\lambda) &= \text{number of crosses} \\ d(\lambda) &= \text{number of cups} \\ k(\lambda) &= rk(\lambda) + d(\lambda). \end{aligned}$$

A bipartition is $(n|n)$ -cross if and only if $k(\lambda) \leq n$. By [BS12b] the modules $R(\lambda^L, \lambda^R)$ have irreducible socle and cosocle equal to $L(\lambda^\dagger)$ where the highest weight λ^\dagger can be obtained by a combinatorial algorithm from λ . Let $\theta : \Lambda \rightarrow X^+(n)$ denote

the resulting map $\lambda \mapsto \lambda^\dagger$ between the set of $(n|n)$ -cross bipartitions Λ and the set $X^+(n)$ of highest weights of \mathcal{R}_n .

Theorem 2.1. [Hei14] *$R = R(\lambda^L, \lambda^R)$ is an indecomposable module of Loewy length $2d(\lambda) + 1$. It is projective if and only if $k(\lambda) = n$ in which case we have $R = P(\lambda^\dagger)$.*

Deligne's interpolating category. For every $\delta \in k$ we dispose over the category $\text{Rep}(Gl_\delta)$ defined in [Del07]. This is a k -linear pseudoabelian rigid tensor category. By construction it contains an object st of dimension δ , called the standard representation. We have a tensor functor $F_n = F_{n|n} : \text{Rep}(Gl_0) \rightarrow \mathcal{R}_n$ by mapping the standard representation of $\text{Rep}(Gl_0)$ to the standard representation of $Gl(n|n)$ in \mathcal{R}_n . Every mixed tensor is in the image of this tensor functor [CW11].

3. THE SYMMETRIC AND ALTERNATING POWERS

We define as in [Hei14]

$$\mathbb{A}_{S^i} := R(i; 1^i) \text{ and } \mathbb{A}_{\Lambda^i} := (\mathbb{A}_{S^i})^\vee = R(1^i; i).$$

We define $S^i = [i, 0, \dots, 0]$ for integers $i \geq 1$. We denote the trivial representation S^0 by $\mathbf{1}$.

Lemma 3.1. [Hei14] *The Loewy structure of the \mathbb{A}_{S^i} is given by ($n \geq 2$)*

$$\begin{aligned} \mathbb{A}_{S^1} &= (\mathbf{1}, S^1, \mathbf{1}) \\ \mathbb{A}_{S^i} &= (S^{i-1}, S^i \oplus S^{i-2}, S^{i-1}) \quad 1 < i \neq n \\ \mathbb{A}_{S^n} &= (S^{n-1}, S^n \oplus S^{n-2} \oplus B^{-1}, S^{n-1}). \end{aligned}$$

We now derive a closed formula for the tensor products $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ and $\mathbb{A}_{\Lambda^i} \otimes \mathbb{A}_{\Lambda^j}$. It turns out that the maximal atypical summands are not irreducible whereas all other summands are irreducible. Therefore we split the computations in two parts: we first compute the projection to the maximal atypical block of $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ and deal with the remaining easy case later in section 6. In the following formulas we often project to the maximal atypical block. Recall from [Hei14] that a mixed tensor $R(\lambda^L, \lambda^R)$ is maximal atypical if and only if $\lambda^R = (\lambda^L)^*$. In this case we simply use the notation $R(\lambda^L)$, e.g. $\mathbb{A}_{S^i} = R(i)$ and $\mathbb{A}_{\Lambda^i} = R(1^i)$.

Lemma 3.2. *The atypical $Gl(1|1)$ -modules in MT are the \mathbb{A}_{S^i} and their duals \mathbb{A}_{Λ^j} . They are the projective covers $\mathbb{A}_{S^i} = P[i-1]$ and $\mathbb{A}_{\Lambda^j} = P[-j+1]$.*

Proof. They are projective since $k(\lambda) = 1$. The statement about the top follows from an explicit computation of the map θ . \square

Corollary 3.3. *In $Gl(1|1)$*

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j} &= \mathbb{A}_{S^{|-i+j|+2}} \oplus 2\mathbb{A}_{S^{|-i+j|+1}} \oplus \mathbb{A}_{S^{|-i+j|}} \\ \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} &= \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \end{aligned}$$

Proof. This is just rewriting the known formula ($a, b \in \mathbf{Z}$)

$$P[a] \otimes P[b] = P[a+b+1] \oplus 2P[a+b] \oplus P[a+b-1]$$

from [GQS07]. \square

Let us assume from now on $m, n \geq 2$.

Lemma 3.4. *After projection to the maximal atypical block ($n \geq 2$)*

$$\begin{aligned}\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j} &= \mathbb{A}_{S|-i+j|+2} \oplus 2\mathbb{A}_{S|-i+j|+1} \oplus \mathbb{A}_{S|-i+j|} \oplus R_1 \\ \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} &= \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \oplus R_2\end{aligned}$$

where R_1 and R_2 are direct sums of modules which do not contain any \mathbb{A}_{S^i} or \mathbb{A}_{Λ^j} .

Proof. This follows from the $Gl(1|1)$ -case and the identification between the projective covers and the symmetric and alternating powers. In $Gl(1|1)$ [GQS07]

$$P[a] \otimes P[b] = P[a+b-1] \oplus 2P[a+b] \oplus P[a+b+1].$$

Hence this formula holds for the corresponding \mathbb{A}_{S^i} respectively \mathbb{A}_{Λ^j} . It then holds in $Rep(Gl_0)$ up to summands in the kernel of $F_1 : Rep(Gl_0) \rightarrow Rep(Gl(1|1))$. The kernel consists of the $R(\lambda)$ with $k(\lambda) > 1$. By [Hei14] a maximal atypical mixed tensor satisfies $d(\lambda) = 1$ (and hence $k(\lambda) = 1$) if and only if and only if $\lambda = (i; 1^i)$ or $\lambda = (1^i; i)$. Hence this formula holds in any $Rep(Gl(n|n))$ up to contributions which lie in the kernel of $F_{n|n} : Rep(Gl_0) \rightarrow Rep(Gl(n|n))$ and which are not $(1|1)$ -cross. \square

In order to compute $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ we compute $R(i) \otimes R(j)$ in $Rep(Gl_0)$. We then push the result to $Rep(Gl(n|n))$ using F_n . We recall the tensor product decomposition in $Rep(Gl_0)$.

Caps. We attach to the weight diagram of a bipartition a cap-diagram as in [BS11]. For integers $i < j$ one says that (i, j) is a $\vee\wedge$ -pair if they are joined by a cap. For $\lambda, \mu \in \Lambda$ one says that μ is linked to λ if there exists an integer $k \geq 0$ and bipartitions $\nu^{(n)}$ for $0 \leq n \leq k$ such that $\nu^{(0)} = \lambda, \nu^{(k)} = \mu$ and the weight diagram of $\nu^{(n)}$ is obtained from the one of $\nu^{(n-1)}$ by swapping the labels of some pair $\vee\wedge$ -pair. Then put

$$D_{\lambda, \mu} = \begin{cases} 1 & \mu \text{ is linked to } \lambda \\ 0 & \text{otherwise.} \end{cases}$$

One has $D_{\lambda, \lambda} = 1$ for all λ . Further $D_{\lambda, \mu} = 0$ unless $\mu = \lambda$ or $|\mu| = (|\lambda^L| - i, |\lambda^R| - i)$ for some $i > 0$. Let t be an indeterminate and R_δ respective R_t the Grothendieck rings of $Rep(GL_\delta)$ over k respective of $Rep(GL_t)$ over $k(t)$. We follow the notation of [CW11] and denote by (λ) the image of $R(\lambda)$ in R_t . Now define $lift_\delta : R_\delta \rightarrow R_t$ as the \mathbf{Z} -linear map defined by $lift_\delta(\lambda) = \sum_\mu D_{\lambda, \mu} \mu$ where the sum runs over all bipartitions μ . By [CW11], Thm. 6.2.3, $lift_\delta$ is a ring isomorphism for every $\delta \in k$.

Tensor products. By [CW11], Thm 7.1.1, the following decomposition holds for arbitrary bipartitions in R_t :

$$\lambda\mu = \sum_{\nu \in \Lambda} \Gamma_{\lambda\mu}^\nu \nu$$

with the numbers

$$\Gamma_{\lambda\mu}^\nu = \sum_{\alpha, \beta, \eta, \theta \in P} \left(\sum_{\kappa \in P} c_{\kappa\alpha}^{\lambda^L} c_{\kappa\beta}^{\mu^R} \right) \left(\sum_{\gamma \in P} c_{\gamma\eta}^{\lambda^R} c_{\gamma\theta}^{\mu^L} \right) c_{\alpha\theta}^{\nu^L} c_{\beta\eta}^{\nu^R},$$

see [CW11], Thm 5.1.2. In particular if $\lambda \vdash (r, s)$, $\mu \vdash (r', s')$, then $\Gamma_{\lambda\mu}^\nu = 0$ unless $|\nu| \leq (r + r', s + s')$. So to decompose tensor products in $Rep(Gl_\delta)$ apply the following three steps: Determine the image of the lift $lift_\delta(\lambda\mu)$ in R_t , use the formula above and then take $lift_\delta^{-1}$.

Lifts. We continue to use our notation for the maximal atypical case and write (i) instead of $(i; 1^i)$. Clearly $\text{lift}(i) = (i) + (i-1)$, $\text{lift}(1^i) = (1^i) + (1^{i-1})$. Hence in order to compute the tensor product $R(i) \otimes R(j)$ we have to compute the tensor product $(i) \otimes (j) \oplus (i) \otimes (j-1) \oplus (i-1) \otimes (j) \oplus (i-1) \otimes (j-1)$ in R_t .

We derive first a closed formula for $(i) \otimes (j)$ in R_t , i.e. $((i, 0, \dots), (1^i)) \otimes (j, 0, \dots), (1^j)$.

- The contribution $\sum_{\gamma \in P} c_{\alpha, \theta}^{\nu^L} c_{\beta, \eta}^{\nu^R}$: Here $\lambda^R = (1^i)$ and $\mu^L = (j, 0, \dots)$. We need to find all pairs of partitions (a, b) such that $c_{a, b}^{\mu^L}$ is non-zero. We denote this by $(\mu^L)^{-1}$. Now the Pieri rule gives $(\mu^L)^{-1} = (0, j), (1, j-1), \dots, (j-1, 1), (j, 0)$ and $(\lambda^R)^{-1} = (0, 1^i), (1, 1^{i-1}), \dots, (1^i, 0)$. This permits only the pairs $(0, i) \leftrightarrow (0, 1^j)$ and $(1, i-1) \leftrightarrow (1, 1^{j-1})$ (to have same γ).
- The contribution $\sum_{\kappa \in P} c_{\kappa, \alpha}^{\lambda^L} c_{\kappa, \beta}^{\mu^R}$: Here $\mu^R = (1^j)$, $\lambda^L = (i)$. As in the previous case this gives only the possibilities $c_{0, i}^i c_{0, 1^j}^{1^j}$ and $c_{1, i-1}^i c_{1, 1^{j-1}}^{1^j}$.

Hence the sum

$$\sum_{\alpha, \beta, \eta, \theta} \left(\sum_{\kappa \in P} c_{\kappa, \alpha}^{\lambda^L} c_{\kappa, \beta}^{\mu^R} \right) \left(\sum_{\gamma \in P} c_{\gamma, \eta}^{\lambda^R} c_{\gamma, \theta}^{\mu^L} \right)$$

collapses to

$$(c_{0, i}^i c_{0, 1^j}^{1^j} + c_{1, i-1}^i c_{1, 1^{j-1}}^{1^j}) (c_{0, 1^i}^{1^i} c_{0, j}^j + c_{1, 1^{i-1}}^{1^i} c_{1, j-1}^j).$$

This corresponds to the choices

- (A) $\alpha = i, \beta = 1^j$
- (B) $\alpha = i-1, \beta = 1^{j-1}$
- (C) $\eta = 1^i, \theta = j$
- (D) $\eta = 1^{i-1}, \theta = j-1$.

Only for these choices AC, AD, BC, BD can there be a non-vanishing contribution $c_{\alpha, \theta}^{\nu^L} c_{\beta, \eta}^{\nu^R}$. From now on we only consider bipartitions ν with $\nu^L = (\nu^R)^*$ and identify such a bipartition with the partition ν^L .

- The AC-case: $c_{i, j}^{\nu^L} c_{1^j, 1^i}^{\nu^R}$. By the Pieri rule ν^L can be any of $(i+j), (i+j-1, 1), (i+j-2, 2), \dots$ and ν^R any of $(1^{i+j}), (2, 1^{i+j-2}), \dots, (i, |i-j|)$. Hence the following partitions ν (i.e. bipartitions of the form $(\nu^L; (\nu^L)^*)$) appear with multiplicity 1:

$$(i+j), (i+j-1, 1), \dots, ((\max(i, j), \min(i, j))).$$

- The AD-case: $c_{i, j-1}^{\nu^L} c_{1^j, 1^{i-1}}^{\nu^R}$. Restricting to $\nu^L = (\nu^R)^*$ we obtain

$$\nu \in \{(i+j-1), (i+j-2, 1), \dots, ((\max(i, j), \min(i, j) - 1))\}.$$

- The BC-case: $c_{i-1, j}^{\nu^L} c_{1^{j-1}, 1^i}^{\nu^R}$. Here ν is any of

$$\nu \in \{((i+j-1), (i+j-2, 1)), \dots, ((\max(i, j), \min(i, j) - 1))\}.$$

- The BD-case: $c_{i-1, j-1}^{\nu^L} c_{1^{j-1}, 1^{i-1}}^{\nu^R}$. Here

$$\nu \in \{((i+j-2), (i+j-3, 1)), \dots, ((\max(i-1, j-1), \min(i-1, j-1)))\}.$$

Hence

$$\begin{aligned}
(i) \otimes (j) = & \\
& (i+j) \oplus (i+j-1, 1) \oplus \dots \oplus ((\max(i, j), \min(i, j))) \\
& \oplus (i+j-1) \oplus (i+j-2, 1) \oplus \dots \oplus ((\max(i, j), \min(i, j) - 1)) \\
& \oplus (i+j-1) \oplus (i+j-2, 1) \oplus \dots \oplus ((\max(i, j), \min(i, j) - 1)) \\
& \oplus ((i+j-2) \oplus (i+j-3, 1) \oplus \dots \oplus (\max(i-1, j-1), \min(i-1, j-1))).
\end{aligned}$$

In the special case $j = 1, i > 1$ we get $(j-1) = 0$ and hence $\text{lift}((i) \otimes (1)) = (i) \otimes (1) \oplus (i) \oplus (i-1) \oplus (i-1) \otimes (1)$. In R_t we have

$$(i) \otimes (1) = (i+1) \oplus (i, 1) \oplus 2(i) \oplus (i-1)$$

so that

$$\text{lift}((i) \otimes (1)) = (i+1) \oplus (i, 1) \oplus 4(i) \oplus (i-1, 1) \oplus 4(i-1) \oplus (i-2).$$

After removing the contributions which will lead to $\mathbb{A}_{S^{i+1}} \oplus 2\mathbb{A}_{S^i} \oplus \mathbb{A}_{S^{i-1}}$ we are left with $(i, 1) \oplus (i) \oplus (i-1, 1) \oplus (i-1)$. Hence

Lemma 3.5. *For $i \geq 2$*

$$\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^1} = \mathbb{A}_{S^{i+1}} \oplus 2\mathbb{A}_{S^i} \oplus \mathbb{A}_{S^{i-1}} \oplus R(i, 1).$$

In the general case we add up the contributions $((i) \oplus (i-1)) \cdot ((j) \oplus (j-1)) = (i)(j) \oplus (i)(j-1) \oplus (i-1)(j) \oplus (i-1)(j-1)$. All the summands are of the following types $(a, 0)$, (a, b) , $a > b > 0$ or (a, a) , $a > 0$. We have

$$\begin{aligned}
\text{lift}(a, b) = & (a, b) \oplus (a, b-1) \oplus (a-1, b) \oplus (a-1, b-1), \quad a > b > 0 \\
\text{lift}(a, a) = & (a, a) \oplus (a, a-1) \oplus (a-1, a-2) \oplus (a-2, a-2).
\end{aligned}$$

After removing the contributions in R_t which will give the $\mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}}$ and applying successively the liftings from above we get the following decompositions. We assume $m = n \geq 2$, $i > j$. For $i > 2, j = 2$ we get

$$\begin{aligned}
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^2} = & \mathbb{A}_{S^{i+2}} \oplus 2 \cdot \mathbb{A}_{S^{i+1}} \oplus \mathbb{A}_{S^i} \\
& \oplus R(i+1, 1) \oplus R(i, 2) \oplus 2 \cdot R(i, 1) \oplus R(i-1, 1)
\end{aligned}$$

Assume now $i > 2, j \geq 2$ und $i \neq j$ and $i > j$. Then

$$\begin{aligned}
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} = & \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \\
& \oplus R(i+j-1, 1) \\
& \oplus R(i+j-2, 2) \oplus 2 \cdot R(i+j-2, 1) \\
& \oplus R(i+j-3, 3) \oplus 2 \cdot R(i+j-3, 2) \oplus R(i+j-3, 1) \\
& \oplus R(i+j-4, 4) \oplus 2 \cdot R(i+j-4, 3) \oplus R(i+j-4, 2) \\
& \oplus R(i+j-5, 5) \oplus \dots \\
& \oplus R(i, j) \oplus 2 \cdot R(i, j-1) \oplus R(i, j-2) \\
& \oplus R(i-1, j-1).
\end{aligned}$$

Now assume $i = j$. For $i = j = 2$ we get

$$\begin{aligned}
\mathbb{A}_{S^2} \otimes \mathbb{A}_{S^2} = & \mathbb{A}_{S^4} \oplus 2 \cdot \mathbb{A}_{S^3} \oplus \mathbb{A}_{S^2} \\
& \oplus R(3, 1) \oplus R(2, 2) \oplus 2 \cdot R(2, 1).
\end{aligned}$$

For $i = j > 2$ we get

$$\begin{aligned}
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} = & \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \\
& \oplus R(i+j-1, 1) \\
& \oplus R(i+j-2, 2) \oplus 2 \cdot R(i+j-2, 1) \\
& \oplus R(i+j-3, 3) \oplus 2 \cdot R(i+j-3, 2) \oplus R(i+j-3, 1) \\
& \oplus R(i+j-4, 4) \oplus 2 \cdot R(i+j-4, 3) \oplus R(i+j-4, 2) \\
& \oplus R(i+j-5, 5) \oplus \dots \\
& \oplus R(i, j) \oplus 2 \cdot R(i, j-1) \oplus R(i, j-2).
\end{aligned}$$

We get the same result as for $i \neq j$ with omitting the last factor $\oplus R(i+j - \min(i, j) - 1, \min(i, j) - 1)$.

Example. We obtain

$$\begin{aligned}
\mathbb{A}_{S^2} \otimes \mathbb{A}_{S^2} &= \mathbb{A}_{S^4} \oplus 2\mathbb{A}_{S^3} \oplus \mathbb{A}_{S^2} \oplus R(3, 1) \oplus R(2, 2) \oplus 2 \cdot R(2, 1) \\
\mathbb{A}_{S^3} \otimes \mathbb{A}_{S^2} &= \mathbb{A}_{S^5} \oplus 2\mathbb{A}_{S^4} \oplus \mathbb{A}_{S^3} \oplus R(4, 1) \oplus R(3, 2) \oplus 2 \cdot R(3, 1) \oplus R(2, 1).
\end{aligned}$$

The highest weights appearing in the socle and head of these indecomposable modules are $[3, 0, \dots, 0]$ (for $\lambda = (4, 1)$), $[2, 1, 0, \dots, 0]$ for $\lambda = (3, 2)$, $[2, 0, \dots, 0]$ for $\lambda = (3, 1)$, $[0, 0, \dots, 0]$ for $\lambda = (2, 2)$ and $[1, 0, \dots, 0]$ for $\lambda = (2, 1)$

4. THE TENSOR PRODUCTS $\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j}$

We derive a closed formula for projection on the maximal atypical block of the tensor product $\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j}$. This won't be needed for the $Gl(2|2)$ calculations. We have

$$lift((i) \otimes (1^j)) = (i) \otimes (1^j) \oplus (i-1) \otimes (1^j) \oplus (i) \otimes (1^{j-1}) \oplus (i-1) \otimes (1^{j-1})$$

in the Grothendieck ring R_t . We may assume that $j > 1$ since $\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^1} = \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^1}$. We may also assume that $i \geq j$ since $(\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j})^\vee = \mathbb{A}_{\Lambda^i} \otimes \mathbb{A}_{S^j}$. We compute $(i) \otimes (1^j)$ in R_t . Recall the classical Pieri rule $(i) \otimes (1^j) = (i+1, 1^{j-1}) \oplus (i, 1^j)$.

- The sum $\sum_{\gamma \in P} c_{\gamma, \eta}^{\lambda^R} c_{\gamma, \theta}^{\mu^L}$: We evaluate this for $\lambda^R = (1^i)$, $\mu^L = (1^j)$. $(\lambda^R)^{-1} = (0, 1^i)$, $(1, 1^{i-1})$, \dots , $(1^i, 0)$ and $(\mu^L)^{-1} = (0, 1^j)$, $(1, 1^{j-1})$, \dots , $(1^j, 0)$. Pairs with the same γ are

$$\begin{aligned}
(0, 1^i) &\leftrightarrow (0, 1^j), \\
(1, 1^{i-1}) &\leftrightarrow (1, 1^{j-1}), \\
&\dots, \\
(1^{\min(i, j)}, 1^{i-|i-j|}) &\leftrightarrow (1^{\min(i, j)}, 1^{j-|i-j|}).
\end{aligned}$$

- The sum $\sum_{\kappa \in P} c_{\kappa, \alpha}^{\lambda^L} c_{\kappa, \beta}^{\mu^R}$: Here $\mu^R = (j)$, $\lambda^L = (i)$. Here the permitted pairs are the

$$\begin{aligned}
(0, i) &\leftrightarrow (0, j), \\
(1, i-1) &\leftrightarrow (1, j-1), \\
&\dots, \\
(\min(i, j), i-|i-j|) &\leftrightarrow (\min(i, j), j-|i-j|).
\end{aligned}$$

Hence the sum $\sum_{\alpha,\beta,\eta,\theta} (\sum_{\kappa \in P} c_{\kappa,\alpha}^{\lambda^L} c_{\kappa,\beta}^{\mu^R}) (\sum_{\gamma \in P} c_{\gamma,\eta}^{\lambda^R} c_{\gamma,\theta}^{\mu^L})$ collapses to

$$(c_{0,i}^i c_{0,j}^j + \dots + c_{\min(i,j),i-|i-j|}^i c_{\min(i,j),j-|i-j|}^j) \\ \cdot (c_{0,1}^1 c_{0,1}^j + \dots + c_{\min(i,j),1^{i-|i-j|}}^1 c_{\min(i,j),1^{j-|i-j|}}^j)$$

We have to evaluate $\sum_{\nu} \sum_{\alpha,\beta,\eta,\theta} c_{\alpha,\theta}^{\nu^L} c_{\beta,\eta}^{\nu^R} \nu$. The following values for these for $\alpha, \beta, \eta, \theta$ give non-vanishing coefficients (let $t = \min(i, j)$):

a) $\alpha = i, \beta = j$	a)' $\eta = 1^i, \theta = 1^j$
b) $\alpha = i-1, \beta = j-1$	b)' $\eta = 1^{i-1}, \theta = 1^{j-1}$
...	...
t) $\alpha = i-t, \beta = j-t$	t)' $\eta = 1^{i-t}, \theta = 1^{j-t}$

This gives $(t+1)^2$ non-vanishing products, namely $aa', ab', \dots, at, ba', bb', \dots, tt$. Now we use $(i) \otimes (1^j) = (i+1, 1^{j-1}) \oplus (i, 1^j)$ in order so see which ones will give maximally atypical ν . Now $\Gamma_{\lambda,\mu}^{\nu} = \sum_{\alpha,\beta,\theta,\eta} \dots = 0$ unless the indices form one of the tuples $aa', ab', \dots, at', ba', bb', \dots, tt'$.

We first treat the partial sum $aa' + ab' + \dots at'$. In that case only aa' and ab' give a contribution. aa' yields $(i+1, 1^{j-1})$ and $(i, 1^j)$ and ab' yields $(i, 1^{j-1})$. Now consider a generic summand $lk', i \neq a, t$. The corresponding product of the Littlewood-Richardson coefficients is $c_{i-l, 1^{j-k}}^{\nu^L} c_{j-k, 1^{i-l}}^{\nu^R}$. The possible ν^L are of the form

$$\nu_1^L = (i-l+1, 1^{j-k-1}), \quad \nu_2^L = (i-l, 1^{j-k})$$

and the possible ν^R are of the form

$$\nu_1^R = (j-k+1, 1^{i-l-1}), \quad \nu_2^R = (j-k, 1^{i-l}).$$

We only consider ν with $\nu^R = (\nu^L)^*$. We have $(\nu_1^L)^* = (j-k, 1^{i-l})$. This is equal to one of the two ν^R for $k=l$ in which case we get (ν_1^L) and (ν_2^L) as a contribution. The pair lk will not give any contribution for $k \notin \{l-1, l, l+1\}$. For $l=k+1$ we get the contribution (ν_1^L) and for $l=k-1$ we get the contribution (ν_2^L) . The sum $ta' + \dots + tt'$ gives the contribution

$$\begin{cases} (i-j+1) \oplus (i-j) & i > j \\ (1^{j-i+1}) \oplus (1^{j-i}) & j > i \\ (1) \oplus (0) & i = j \end{cases}$$

Hence we obtain the following closed formula:

$$(i) \otimes (1^j) = (i+1, 1^{j-1}) \oplus (i, 1^j) \oplus (i, 1^{j-1}) \\ \oplus \bigoplus_{l=1}^{t-1} [(i-l, 1^{j-l-1}) \oplus (i-l, 1^{j-l}) \oplus (i-l+1, 1^{j-l-1}) \oplus (i-l+1, 1^{j-l})] \\ \oplus \begin{cases} (i-j+1) \oplus (i-j) & i > j \\ (1^{j-i+1}) \oplus (1^{j-i}) & j > i \\ (1) \oplus (0) & i = j \end{cases}$$

We apply this formula to the four summands of $\text{lift}((i) \otimes (1^j))$, namely $(i) \otimes (1^j)$, $(i-1) \otimes (1^j)$, $(i) \otimes (1^{j-1})$ and $(i-1) \otimes (1^{j-1})$. The contributions in the total sum

are either of the form (i) or (1^j) or $(i, 1^j)$. We have

$$lift(i, 1^j) = (i, 1^j) \oplus (i-1, 1^j) \oplus (i, 1^{j-1}) \oplus (i-1, 1^{j-1}).$$

From the $Gl(1|1)$ -case (see 3.4) we know that the contribution of the alternating and symmetric powers will be given by $(i > j)$

$$\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j} = \mathbb{A}_{S|-i+j|+2} \oplus 2\mathbb{A}_{S|-i+j|+1} \oplus \mathbb{A}_{S|-i+j|} \oplus R$$

and by

$$\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^i} = \mathbb{A}_{S+2} \oplus 2\mathbb{A} \oplus \mathbb{A}_{\Lambda^2} \oplus R$$

for $i = j$ for some R -term which does not involve any alternating or symmetric powers. Removing all the corresponding bipartitions from the total sum and working downwards as in the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ -case we obtain the final result. For $i = j = 2$ we obtain:

$$\mathbb{A}_{S^2} \otimes \mathbb{A}_{\Lambda^2} = \mathbb{A}_{S+2} \oplus 2\mathbb{A} \oplus \mathbb{A}_{\Lambda^2} \oplus R(3, 1) \oplus R(2, 1^2) \oplus 2R(2, 1)$$

and for $i > j = 2$ we obtain

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^2} = \\ \mathbb{A}_{S^i} \oplus 2\mathbb{A}_{S^{i-1}} \oplus \mathbb{A}_{S^{i-2}} \oplus R(i+1, 1) \oplus R(i, 1^2) \oplus 2R(i, 1) \oplus R(i-1, 1). \end{aligned}$$

The general formula is for $i > j > 2$ as follows

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j} = & \mathbb{A}_{S|-i+j|+2} \oplus 2\mathbb{A}_{S|-i+j|+1} \oplus \mathbb{A}_{S|-i+j|} \\ & \oplus R(i+j-(j-1), 1^{j-1}) \\ & \oplus R(i+j-j, 1^j) \oplus 2\dot{R}(i, 1^{j-1}) \oplus R(i, 1^{j-2}) \\ & \dots \\ & \oplus R(i+j-k, 1^k) \oplus 2 \cdot R(i+j-k, 1^{k-1}) \oplus R(i+j-k, 1^{k-2}) \\ & \oplus \dots \\ & \oplus R(i-j+2, 1^2) \oplus 2 \cdot R(i-j+2, 1) \\ & \oplus R(i-j+1, 1). \end{aligned}$$

For $i = j > 2$ one has to remove the last term $R(i-j+1, 1)$.

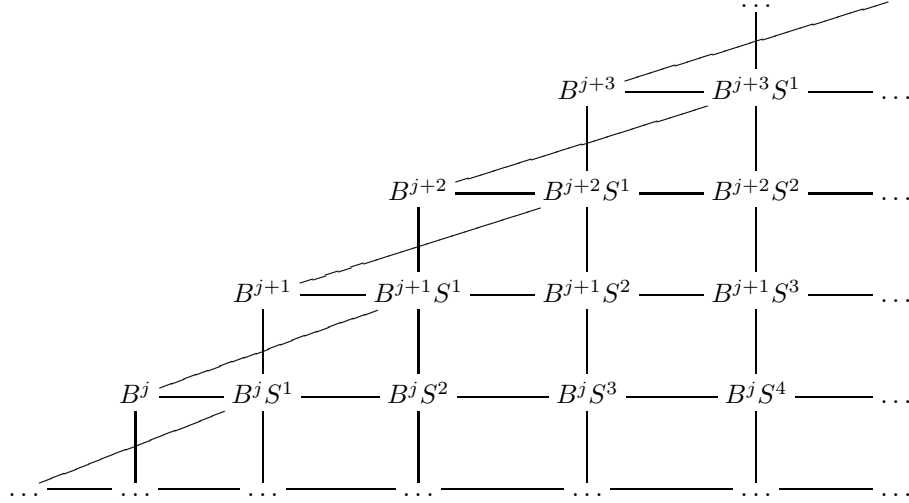
Example. For any $n \geq 2$ we get

$$\begin{aligned} \mathbb{A}_{S^3} \otimes \mathbb{A}_{\Lambda^2} &= \mathbb{A}_{S^3} \oplus 2\mathbb{A}_{S^2} \oplus \mathbb{A}_{S^1} \oplus R(4, 1) \oplus R(3, 1^2) \oplus 2R(3, 1) \oplus R(2, 1) \\ \mathbb{A}_{S^3} \otimes \mathbb{A}_{\Lambda^3} &= \mathbb{A}_{S^2} \oplus 2\mathbb{A} \oplus \mathbb{A}_{\Lambda^2} \oplus R(4, 1^2) \oplus R(3, 1^3) \oplus 2R(3, 1^2) \oplus R(3, 1) \\ &\quad \oplus R(2, 1^2) \oplus 2R(2, 1). \end{aligned}$$

5. $Gl(2|2)$ TENSOR PRODUCTS - THE MAXIMAL ATYPICAL PART

We compute the decomposition of the tensor product of any two maximally atypical irreducible modules in \mathcal{R}_2 . In this section we compute only the direct summands which are maximal atypical. The remaining summands are computed in section 6. The basic idea is to look at our formulas for $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ in the Grothendieck group and use these to compute the composition factors of $S^i \otimes S^j$ recursively starting with the obvious tensor product $S^i \otimes S^0$. We then determine the decomposition into indecomposable summands using results on cohomological tensor functors [HW14] and case-by-case distinctions.

5.1. The \mathcal{R}_2 -case: Setup. Every maximally atypical irreducible representation $L(\lambda) = [\lambda_1, \lambda_2]$ is a Berezin twist of a representation of the form $S^i := [i, 0]$ for $i \in \mathbb{N}$. Since tensoring with Ber is a flat functor, it is therefore enough to decompose the tensor product $S^i \otimes S^j$. The Ext-quiver of the maximal atypical block Γ of \mathcal{R}_2 can be easily determined from [BS10a]. It has been worked out by [Dro09]. For all irreducible modules in Γ we have $\dim \text{Ext}^1(L(\lambda), L(\mu)) = \dim \text{Ext}^1(L(\mu), L(\lambda)) = 0$ or 1. The Ext-quiver can be pictured as follows where a line segment between two irreducible modules denotes a non-trivial extension class between these two modules and where an irreducible module $[x, y]$ is represented as a point in \mathbf{Z}^2 .



The Loewy structure of the projective covers of a maximally atypical irreducible module can also be computed from [BS12a] or be taken from Drouot: For $[a, b]$, $a = b + k$, $k \geq 3$ the Loewy structure (we display the socle layers) is

$$P[a, b] = \begin{pmatrix} B^{a-k} S^k \\ B^{a-k} S^{k+1} \oplus B^{a-k} S^{k-1} \oplus B^{a-k-1} S^{k+1} \oplus B^{a-k+1} S^{k-1} \\ 2B^{a-k} S^k \oplus B^{a-k-1} S^{k+2} \oplus B^{a-k-1} S^k \oplus B^{a-k+2} S^{k-3} \\ B^{a-k} S^{k+1} \oplus B^{a-k} S^{k-1} \oplus B^{a-k-1} S^{k+1} \oplus B^{a-k+1} S^{k-1} \\ B^{a-k} S^k \end{pmatrix}.$$

For $[a, b]$, $a = b + 2$ the Loewy structure is

$$P[a, b] = \begin{pmatrix} B^{a-2} S^2 \\ B^{a-2} S^3 \oplus B^{a-2} S^1 \oplus B^{a-3} S^3 \oplus B^{a-1} S^1 \\ 2B^{a-2} S^2 \oplus B^{a-3} S^4 \oplus B^{a-3} S^2 \oplus B^{a-1} S^2 \oplus B^{a-1} \oplus B^{a-2} \\ B^{a-2} S^3 \oplus B^{a-2} S^1 \oplus B^{a-3} S^3 \oplus B^{a-1} S^1 \\ B^{a-2} S^2 \end{pmatrix}.$$

For $[a, b]$, $a = b + 1$ the Loewy structure is

$$P[a, b] = \begin{pmatrix} B^{a-1} S^1 \\ B^{a-1} S^2 \oplus B^{a-1} \oplus B^{a-2} S^2 \oplus B^a \text{ plus } B^{a-2} \\ 2B^{a-1} S^1 \oplus B^{a-2} S^3 \oplus B^{a-2} S^1 \oplus B^a S^1 \oplus \\ B^{a-1} S^2 \oplus B^{a-1} \oplus B^{a-2} S^2 \oplus B^a \oplus B^{a-2} \\ B^{a-1} S^1 \end{pmatrix}.$$

For $[a, b], a = b$ the Loewy structure is

$$P[a, b] = \begin{pmatrix} & & B^a \\ & B^a S^1 \oplus B^{a-1} S^1 \oplus B^{a+1} S^1 & \\ 2B^a \oplus B^{a-1} \oplus B^{a-2} \oplus B^{a-1} S^2 \oplus B^a S^2 \oplus B^{a+1} \oplus B^{a+2} & & \\ & B^a S^1 \oplus B^{a-1} S^1 \oplus B^{a+1} S^1 & \\ & & B^a \end{pmatrix}.$$

5.2. The \mathcal{R}_2 -case: Mixed tensors. We specialise our decompositions to the \mathcal{R}_2 -case. All formulas hold only after projection to Γ . For $i, j \leq 2$ we get

$$\begin{aligned} \mathbb{A}_{S^1} \otimes \mathbb{A}_{S^1} &= \mathbb{A}_{S^2} \oplus 2\mathbb{A}_{S^1} \oplus \mathbb{A}_{S^2}^\vee \\ \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^1} &= \mathbb{A}_{S^{i+1}} \oplus 2 \cdot \mathbb{A}_{S^i} \oplus \mathbb{A}_{S^{i-1}} \oplus P[i-1, 0]. \\ \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^2} &= \mathbb{A}_{S^{i+2}} \oplus 2 \cdot \mathbb{A}_{S^{i+1}} \oplus \mathbb{A}_{S^i} \\ &\quad \oplus P([i, 0]) \oplus P([i-1, 1] \oplus 2 \cdot P([i-1, 0]) \oplus P([i-2, 0]) \end{aligned}$$

where we assumed $i > 1$ respectively $i > 2$. Assume now $i > 2, j \geq 2$ und $i \neq j$ and without loss of generality $i > j$.

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} &= \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \\ &\quad \oplus P[i+j-2, 0]) \\ &\quad \oplus P[i+j-3, 1] \oplus 2 \cdot P[i+j-3, 0] \\ &\quad \oplus P[i+j-4, 2] \oplus 2 \cdot P[i+j-4, 1] \oplus P[i+j-4, 0] \\ &\quad \oplus P[i+j-5, 3] \oplus 2 \cdot P[i+j-5, 2] \oplus P[i+j-5, 1] \\ &\quad \oplus P[i+j-6, 4] \oplus \dots \\ &\quad \oplus P[i-1, j-1] \oplus 2 \cdot P[i-1, j-2] \oplus P[i-1, j-3] \\ &\quad \oplus P[i-2, j-2]. \end{aligned}$$

For $i = j = 2$

$$\mathbb{A}_{S^2} \otimes \mathbb{A}_{S^2} = \mathbb{A}_{S^4} \oplus 2\mathbb{A}_{S^3} \oplus \mathbb{A}_{S^2} \oplus P[2, 0] \oplus P[0, 0] \oplus 2P[1, 0].$$

For $i = j > 2$ we have

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} &= \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \\ &\quad \oplus P[i+j-2, 0]) \\ &\quad \oplus P[i+j-3, 1] \oplus 2 \cdot P[i+j-3, 0] \\ &\quad \oplus P[i+j-4, 2] \oplus 2 \cdot P[i+j-4, 1] \oplus P[i+j-4, 0] \\ &\quad \oplus P[i+j-5, 3] \oplus 2 \cdot P[i+j-5, 2] \oplus P[i+j-5, 1] \\ &\quad \oplus P[i+j-6, 4] \oplus \dots \\ &\quad \oplus P[i-2, i-2] \oplus 2 \cdot P[i-1, i-2] \oplus P[i-1, i-3]. \end{aligned}$$

5.3. The \mathcal{R}_2 -case: K_0 -decomposition. The tensor product decomposition of the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ along with the knowledge of the composition factors of the indecomposable summands permits to give recursive formulas for the K_0 -decomposition of the tensor products $S^i \otimes S^j$. Due to the asymmetry of the formulas and the asymmetry of the K_0 -decompositions for \mathbb{A}_{S^i} and $P[a, b]$ for small i and $a - b$ we compute the tensor products for small i and j first. The K_0 -decomposition $S^1 \otimes S^1$ follows

immediately from the $\mathbb{A}_{S^1} \otimes \mathbb{A}_{S^1}$ -decomposition since all other factors are known. We get

$$S^1 \otimes S^1 = 2\mathbf{1} \oplus 2S^1 \oplus B \oplus B^{-1} \oplus B^{-1}S^2 \oplus S^2.$$

Similarly one computes

$$\begin{aligned} S^2 \otimes S^1 &= 2S^2 + S^3 + B^{-1}S^3 + S^1 + BS^1 \\ S^2 \otimes S^2 &= S^4 + B^{-1}S^4 + 2S^3 + S^2 + BS^2 + 2BS^1 + \mathbf{1} + 2B + B^2. \end{aligned}$$

Lemma 5.1. *For $i \geq 1$ we have $P[i, 0] = 2\mathbb{A}_{S^{i+1}} + B^{-1}\mathbb{A}_{S^{i+2}} + B\mathbb{A}_{S^i}$.*

Proof. This is just a direct inspection of the Loewy structures above. \square

Lemma 5.2. *For all $i > j$ we have in the Grothendieck group*

$$\begin{aligned} S^i \otimes S^j &= 2(S^{i+j-1} + \text{Ber}S^{i+j-3} + \dots + \text{Ber}^{j-1}S^{i-j+1}) \\ &\quad + S^{i+j}(1 + \text{Ber}^{-1}) + \dots + \text{Ber}^b S^{i-j}(1 + \text{Ber}^{-1}). \end{aligned}$$

For $i = j$ we get

$$\begin{aligned} S^i \otimes S^i &= 2(S^{2i-1} + \text{Ber}S^{2i-3} + \dots + \text{Ber}^{i-1}S^1) \\ &\quad + S^{2i}(1 + \text{Ber}^{-1}) + \dots + \text{Ber}^i(1 + \text{Ber}^{-1}) + B^{i-1} + B^{i-2}. \end{aligned}$$

Proof. We first consider the cases $S^i \otimes S^1$ and $S^i \otimes S^2$ for $i > 1$ respectively $i > 2$. The case $S^i \otimes S^1$, $i > 1$: For the induction start $i = 2$ see above. Put $C_i = S^i \otimes S^1$ in $K_0(\mathcal{R}_n)$. For $i \geq 4$ we get then the uniform formula $S^i \otimes S^1 + 2S^{i-1} \otimes S^1 + S^{i-2} \otimes S^1 = (S^{i+1} + 2S^i + S^{i-1}) + (S^{i-1} + 2S^{i-2} + S^{i-3}) + (2C_{i-1} + \text{Ber}^{-1}S^{i+1} + \text{Ber}^{-1}S^{i-1} + \text{Ber}S^{i-1} + \text{Ber}S^{i-3})$. Hence using the induction assumption $S^{i-2} \otimes S^1 = 2S^{i-2} + S^{i-1} + \text{Ber}^{-1}S^{i-1} + S^{i-3} + \text{Ber}S^{i-3}$ we get $S^i \otimes S^1 = 2S^i + S^{i+1} + S^{i-1} + \text{Ber}^{-1}S^{i+1} + \text{Ber}S^{i-1}$, and this proves the induction step. Likewise for $S^i \otimes S^2$. Now assume $i > j > 2$. Then for $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ we get the regular formula in $K_0(\mathcal{R}_n)$

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} &= S^i \otimes S^j + 4(S^{i-1} \otimes S^{j-1}) + 2(S^{i-1} \otimes S^j) + \\ &\quad 2(S^{i-1} \otimes S^{j-2}) + 2(S^i \otimes S^{j-1}) + S^i \otimes S^{j-2} + \\ &\quad 2(S^{i-2} \otimes S^{j-1}) + S^{i-2} \otimes S^j + S^{i-2} \otimes S^{j-2}. \end{aligned}$$

All tensor products except $S^i \otimes S^j$ are known by induction. On the other hand this sum equals $\mathbb{A}_{S^{i+j}} + 2\mathbb{A}_{S^{i+j}} + \mathbb{A}_{S^{i+j-2}} + P[i+j-2, 0] + 2P[i+j-3, 0] + P[i+j-4, 0](1+B) + 2BP[i+j-5, 0] + BP[i+j-6, 0](1+B) + \dots + 2B^{j-2}P[i-j+1, 0] + B^{j-2}P[i-j, 0](1+B)$. Plugging in $P[a, 0] = 2\mathbb{A}_{S^{a+1}} + B^{-1}\mathbb{A}_{S^{a+2}} + \mathbb{A}_{S^a}$ for all $a \geq 1$ and comparing terms with the same B -power on both sides finishes the proof. The case $i = j$ works exactly the same way. \square

5.4. The \mathcal{R}_2 -case: Socle Estimates. We say $w(M) = k$ for a module M , if $M^\vee \cong \text{Ber}^{-k}M$. Examples: $w(S^i) = i - 1$ and $w(\text{Ber}) = 2$, and therefore

$$w(S^i \otimes S^j) = i + j - 2.$$

On the other hand for $*$ -selfdual modules M we have

$$\text{soc}(M) \cong \text{cosoc}(M),$$

since $*$ -duality is trivial on semisimple modules. On the other hand $w(M) = k$ implies $\text{soc}(M)^\vee \cong \text{Ber}^{-k} \text{cosoc}(M)$, so that both conditions together imply $w(\text{soc}(M)) = k$. Hence being semi-simple, it is a direct sum of modules

$$\text{soc}(M) \cong \text{soc}'(M) \oplus \bigoplus_{\nu \in \mathbf{Z}} m(\nu) \cdot \text{Ber}^\nu S^{k+1-2\nu}$$

with $S^i = 0$ for $i < 0$ and certain multiplicities $m(\nu)$, plus a sum $\text{soc}'(M)$ of modules of type

$$(\text{Ber}^\nu \oplus \text{Ber}^{k-\nu-j+1}) S^j$$

for certain $\nu \in \mathbf{Z}$ and certain natural numbers j with $k - \nu - j + 1 \neq \nu$.

Proposition 5.3. *For $i > j \geq 2$ we have $\text{soc}'(M) = 0$ for $M = S^{i-1} \otimes S^{j-1}$ and*

$$\text{soc}(S^{i-1} \otimes S^{j-1}) \hookrightarrow 3 \cdot S^{i+j-3} \oplus 2 \cdot \text{Ber} S^{i+j-5} \oplus \dots \oplus 2 \cdot \text{Ber}^{j-2} S^{i-j+1}.$$

For $i = j \geq 2$ we have

$$\text{soc}(S^{i-1} \otimes S^{i-1}) \hookrightarrow 3 \cdot S^{2i-3} \oplus 2 \cdot \text{Ber} S^{2i-5} \oplus \dots \oplus 2 \cdot \text{Ber}^{i-2} S^1 \oplus B^{i-4}.$$

Proof. Assume $i > j$. Note that $\text{soc}(M) \hookrightarrow \text{soc}(\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j})$ and by the above formulas the latter is

$$\begin{aligned} & S^{i+j-1} \oplus 3S^{i+j-2} \oplus 3S^{i+j-3} \oplus (\text{Ber} \oplus \mathbf{1}) S^{i+j-4} \oplus 2\text{Ber} S^{i+j-5} \\ & \oplus (\text{Ber} \oplus \mathbf{1}) \text{Ber} S^{i+j-6} \oplus 2\text{Ber}^2 S^{i+j-7} \oplus \dots \\ & \oplus (\text{Ber} \oplus \mathbf{1}) \text{Ber}^{j-2} S^{i-j} \oplus 2\text{Ber}^{j-2} S^{i-j+1}. \end{aligned}$$

Since $k = w(M) = (i-1) - 1 + (j-1) - 1 = i+j-4$, this implies the assertion $\text{soc}'(M) = 0$. Indeed the terms $S^{i+j-1} \oplus 3S^{i+j-2}$ and also $N = (\text{Ber} \oplus \mathbf{1}) \text{Ber}^\nu S^{i+j-4-2\nu}$ cannot contribute to $\text{soc}'(M)$, since

$$\begin{aligned} N^\vee &= (\text{Ber}^{-1} \oplus \mathbf{1}) \text{Ber}^{-\nu} \text{Ber}^{-i-j+3+2\nu} S^{i+j-4-2\nu} \\ &= (\text{Ber}^{-1} \oplus \mathbf{1}) \text{Ber}^{-i-j+3+\nu} S^{i+j-4-2\nu} \end{aligned}$$

and

$$\begin{aligned} \text{Ber}^{-k} N &= \text{Ber}^{-k} (\text{Ber} \oplus \mathbf{1}) \text{Ber}^\nu S^{i+j-4-2\nu} \\ &= (\text{Ber}^2 \oplus \text{Ber}) \text{Ber}^{-i-j+3+\nu} S^{i+j-4-2\nu} \end{aligned}$$

have no common irreducible summand. Hence $\text{soc}(M)$ is contained in $3 \cdot S^{i+j-3} \oplus 2 \cdot \text{Ber} S^{i+j-5} \oplus \dots \oplus 2 \cdot \text{Ber}^{j-2} S^{i-j+1}$. The proof is analogous for $i = j$. \square

5.5. The Duflo-Serganova functor DS . We recall some constructions from the preprint [HW14].

An embedding. We view $G_{n-1} = \text{Gl}(n-1|n-1)$ as an ‘outer block matrix’ in $G_n = \text{Gl}(n|n)$ and G_1 as the ‘inner block matrix’ at the matrix positions $n \leq i, j \leq n+1$. Fix the following element $x \in \mathfrak{g}_1$,

$$x = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \text{ for } y = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We furthermore fix the embedding

$$\varphi_{n,1} : G_{n-1} \times G_1 \hookrightarrow G_n$$

defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & B \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ C & 0 & 0 & D \end{pmatrix}.$$

We use this embedding to identify elements in G_{n-1} and G_1 with elements in G_n . In this sense $\epsilon_n = \epsilon_{n-1}\epsilon_1$ holds in G_n , for the corresponding elements ϵ_{n-1} and ϵ_1 in G_{n-1} resp. G_1 , defined in section 1.

Two functors. One has a functor $(V, \rho) \mapsto V^+ = \{v \in V \mid \rho(\epsilon_1)(v) = v\}$

$$^+ : \mathcal{R}_n \rightarrow \mathcal{R}_{n-1}$$

where V^+ is considered as a G_{n-1} -module using $\rho(\epsilon_1)\rho(g) = \rho(g)\rho(\epsilon_1)$

Similarly define $V^- = \{v \in V \mid \rho(\epsilon_1)(v) = -v\}$. With the grading induced from $V = V_0 \oplus V_1$ this defines a representation V^- of G_{n-1} in $\Pi\mathcal{R}_{n-1}$. Obviously

$$(V, \rho)|_{G_{n-1}} = V^+ \oplus V^-.$$

Cohomological tensor functors. Since x is an odd element with $[x, x] = 0$, we get

$$2 \cdot \rho(x)^2 = [\rho(x), \rho(x)] = \rho([x, x]) = 0$$

for any representation (V, ρ) of G_n in \mathcal{R}_n . Notice $d = \rho(x)$ supercommutes with $\rho(G_{n-1})$. Furthermore $\rho(x) : V^\pm \rightarrow V^\mp$ holds as a k -linear map, an immediate consequence of $d\rho(\epsilon_1) = -\rho(\epsilon_1)d$, i.e. of $Ad(\epsilon_1)(x) = -x$. Since $\rho(x)$ is an *odd* morphism, $\rho(x)$ induces the following *even* morphisms (morphisms in \mathcal{R}_{n-1})

$$\rho(x) : V^+ \rightarrow \Pi(V^-) \quad \text{and} \quad \rho(x) : \Pi(V^-) \rightarrow V^+.$$

The k -linear map $\partial = \rho(x) : V \rightarrow V$ is a differential and commutes with the action of G_{n-1} on (V, ρ) . Therefore ∂ defines a complex in \mathcal{R}_{n-1}

$$\xrightarrow{\partial} V^+ \xrightarrow{\partial} \Pi(V^-) \xrightarrow{\partial} V^+ \xrightarrow{\partial} \dots$$

Since this complex is periodic, it has essentially only two cohomology groups denoted $H^+(V, \rho)$ and $H^-(V, \rho)$ in the following. This defines two functors $(V, \rho) \mapsto D_{n,n-1}^\pm(V, \rho) = H^\pm(V, \rho)$

$$D_{n,n-1}^\pm : \mathcal{R}_n \rightarrow \mathcal{R}_{n-1}.$$

For the categories $T = T_n$ resp. T_{n-1} (for the groups G_n resp. G_{n-1}) consider the tensor functor of Duflo and Serganova in [DS05]

$$DS_{n,n-1} : T_n \rightarrow T_{n-1}$$

defined by $DS_{n,n-1}(V, \rho) = V_x := \text{Kern}(\rho(x))/\text{Im}(\rho(x))$. Then for $(V, \rho) \in \mathcal{R}_n$

$$H^+(V, \rho) \oplus \Pi(H^-(V, \rho)) = DS_{n,n-1}(V).$$

Indeed, the left side is $DS_{n,n-1}(V) = V_x$ for the k -linear map $\partial = \rho(x)$ on $V = V^+ \oplus V^-$. Hence H^+ is the functor obtained by composing the tensor functor

$$DS_{n,n-1} : \mathcal{R}_n \rightarrow T_{n-1}$$

with the functor

$$T_{n-1} \rightarrow \mathcal{R}_{n-1}$$

The ring homomorphism d . As an element of the Grothendieck group $K_0(\mathcal{R}_{n-1})$ we define for a module $M \in \mathcal{R}_n$

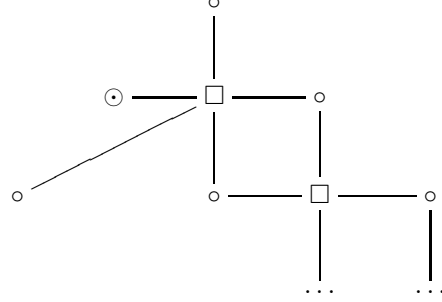
The map d is additive by [HW14]. Notice

We have a commutative diagram

where the horizontal maps are surjective ring homomorphisms defined by

Since DS induces a ring homomorphism, d defines a ring homomorphism.

with the two \circ to the upper left at position $B^j S^{i-j}$ and $B^{j-1} S^{i-j}$ and the ones to the lower right at position $B^{-1} S^{i+j}$ and S^{i+j} . The picture in the $i = j$ -case is similar



with the composition factor \odot at position B^{i-1} appearing with multiplicity 2 and the additional \circ at position B^{i-2} .

We now make use of the cohomological tensor functors DS . In the $Gl(1|1)$ -case $S^i \simeq B^i$ and hence $S^i \otimes S^j = S^{i+j}$. We know from [HW14] that $DS(S^i) = S^i + B^{-1}[1-i]$ and $DS(B) = B[-1]$. Hence $DS(S^i \otimes S^j)$ splits into four indecomposable summands each of superdimension 1 or each of superdimension -1:

$$\begin{aligned} DS(S^i \otimes S^j) &= (S^i \oplus B^{-1}) \otimes (S^j \oplus B^{-1}[1-j]) \\ &= B^{i+j} \oplus B^{i-1}[1-j] \oplus B^{j-1}[1-i] \oplus B^{-2}[2-i-j]. \end{aligned}$$

Hence $M = S^i \otimes S^j$ splits into at most four indecomposable summands of $sdim \neq 0$.

Lemma 5.4. *Every atypical direct summand is $*$ -invariant.*

Proof. If I is a direct summand which is not $*$ -invariant, M contains I^* as a direct summand and $[I] = [I^*]$ in $K_0(\mathcal{R}_n)$. However any summand of length > 1 must contain a factor of type \circ which occur in M only with multiplicity 1, a contradiction. \square

Corollary 5.5. *The superdimension of any maximally atypical summand is $\neq 0$.*

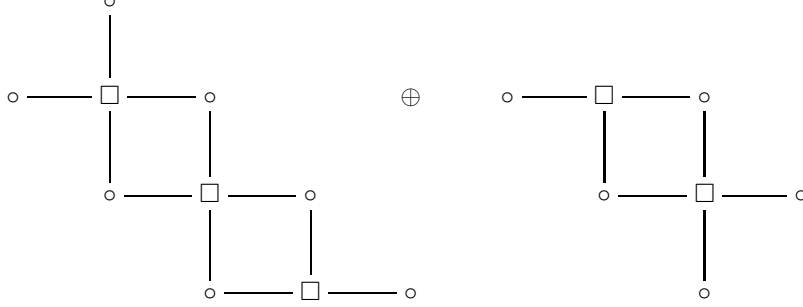
Proof. M does not contain any projective cover (look at composition factors). If $sdim(I) = 0$, $DS(I) = 0$. However $\ker(DS) = AKac$ [HW14] (the modules with a filtration by AntiKac-modules) which are not $*$ -invariant, unless they are projective. \square

Assume $i > j$. By $*$ -invariance the Loewy length of a direct summand is either 1 or 3. If I is irreducible, then necessarily $I = \square$ for a composition factor of the socle. By socle considerations both \square will split as direct summands. The remaining module has superdimension zero, hence the Loewy length of a direct summand is 3. Fix a composition factor of type \square . The multiplicity of \square in the socle cannot be 2. If the multiplicity of \square in the socle is zero, then \square has to be in the middle Loewy layer. But this would force composition factors of type \circ to be in the socle. Contradiction. Hence

Corollary 5.6. *For $i > j$*

$$soc(S^i \otimes S^j) = S^{i+j-1} \oplus BerS^{i+j-3} \oplus \dots \oplus Ber^{j-1}S^{i-j+1}.$$

Assume $i > j$. Then the superdimension of a direct summand is either 2 or 4. Hence M is either indecomposable or splits into two summands $M = I_1 \oplus I_2$ of superdimension 2. If M would split, it would split in the following way:



Now we use the ring homomorphism $d(M) = H^+(M) - H^-(M)$, $d : K_0(\mathcal{R}_n) \rightarrow K_0(\mathcal{R}_{n-1})$ defined above. We know

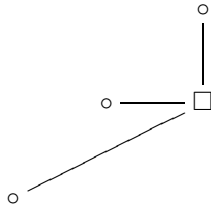
$$d(S^i \otimes S^j) = B^{i+j} + (-1)^{1-j} B^{i-1} + (-1)^{1-i} B^{j-1} + (-1)^{2-i-j} B^{-2}.$$

Since DS maps Anti-Kac modules to zero, $d()$ of any square with edges $B^k S^i$, $B^{k+1} S^{i-1}$, $B^{k+1} S^i$, $B^k S^{i+1}$ is zero. Hence $d(I_2)$ is given by applying d to the hook in the lower right $d(S^{i+j} + S^{i+j-1} + B^{-1} S^{i+j})$ and to $(B^v S^{i+j+1-2v} + B^v S^{i+j-2v})$ from the upper left of I_2 . We get $d(I_2) = B^{i+j} + (-1)^{i-j} B^{-2} + (-1)^v B^{i+j+1-v} + (-1)^v B^{i+j-v}$ with the two additional summands $(-1)^v B^{i+j+1-v} + (-1)^v B^{i+j-v}$. Contradiction, hence M is indecomposable.

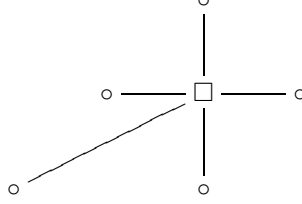
Now assume $i = j$. By the socle estimates for $S^i \otimes S^i$ and $*$ -duality either B^{i-1} splits as a direct summand or both B^{i-1} lie in the middle Loewy layer. Note that $\text{Hom}(B^{i-1}, S^i \otimes S^i) = \text{Hom}(B^{i-1} \otimes (S^i)^\vee, S^i) = \text{End}(S^i) = k$, hence the last case cannot happen. Hence B^{i-1} splits as a direct summand. We show that the remaining module M' in $S^i \otimes S^i = B^{i-1} \oplus M'$ is indecomposable. As in the $i > j$ -case the Loewy length of any direct summand of M' must be 3. As before we obtain for $i = j$

$$\text{soc}(S^i \otimes S^i) = S^{2i-1} \oplus \text{Ber} S^{2i-3} \oplus \dots \oplus \text{Ber}^{i-1} S^1 \oplus B^{i-1}.$$

The remaining part M' can either split into three indecomposable modules of superdimension one each, in a direct sum of two modules of superdimension one respectively two or is indecomposable. One cannot split the upper left \tilde{I}



as a direct summand since its superdimension is -1 . Similarly one cannot split



as a direct summand since the remaining module would have superdimension zero. Since all composition factors except the B 's have superdimension ± 2 , M' could split only into $M' = I_1 \oplus I_2$ with $\text{sdim}(I_1) = 1$ and $\text{sdim}(I_2) = 2$ with I_2 as above. We argue now as in the $i > j$ -case. We have

$$d(M) = B^{2i} + (-1)^{1-i} B^{i-1} + (-1)^{1-i} B^{i-1},$$

but $d(I_2)$ has four summands as in the $i > j$ -case. Contradiction, hence M is indecomposable.

Corollary 5.7. *Up to summands which are not in the maximal atypical block we obtain the following decompositions. $S^i \otimes S^j \simeq M$ ($i > j$) where M is indecomposable with Loewy structure*

$$\begin{pmatrix} S^{i+j-1} \oplus \text{Ber} S^{i+j-3} \oplus \dots \oplus \text{Ber}^{j-1} S^{i-j+1} \\ S^{i+j}(1 + \text{Ber}^{-1}) + \dots + \text{Ber}^j S^{i-j}(1 + \text{Ber}^{-1}) \\ S^{i+j-1} \oplus \text{Ber} S^{i+j-3} \oplus \dots \oplus \text{Ber}^{j-1} S^{i-j+1} \end{pmatrix}$$

and $S^i \otimes S^i = B^{i-1} \oplus M$ where M is indecomposable with Loewy structure

$$\begin{pmatrix} S^{2i-1} \oplus \text{Ber} S^{2i-3} \oplus \dots \oplus \text{Ber}^{i-1} S^1 \\ S^{2i}(1 + \text{Ber}^{-1}) + \dots + \text{Ber}^i S^0(1 + \text{Ber}^{-1}) + B^{i-2} \\ S^{2i-1} \oplus \text{Ber} S^{2i-3} \oplus \dots \oplus \text{Ber}^{i-1} S^1 \end{pmatrix}.$$

6. $GL(2|2)$ TENSOR PRODUCTS - THE GENERAL CASE

We compute the remaining contributions to the tensor product $S^i \otimes S^j$ in \mathcal{R}_n for $n \geq 2$. These are all irreducible which basically follows from the fact that all lower atypical summands in an $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ tensor product are irreducible.

Lemma 6.1. *$\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ is a direct sum of maximally atypical summands and $(n-2)$ -times atypical irreducible representations. Likewise for $\mathbb{A}_{\Lambda^i} \otimes \mathbb{A}_{\Lambda^j}$. The $(n-2)$ -times atypical summands are irreducible.*

Proof. In the decomposition of $\text{lift}((i; 1^i) \otimes (j; 1^j))$ in R_t , the bipartitions which will not contribute to the maximal atypical block are of the form

$$[(i+j-k, k); (2^r, 1^{i+j-2r})]$$

for some $k, r \geq 0$ and $k \neq r$. We have

$$I_{\wedge} = \{i+j-k, k-1, -2, -3, -4, \dots\}$$

$$I_{\vee} = \{-1, 0, 1, \dots, r-2, r, r+1, \dots, i+j-r-1, i+j-r+1, \dots\}$$

Since $k \neq r$, neither one of the two conditions $i+j-k = i+j-r$, $k-1 = r-1$ is satisfied, hence the two sets intersect at two points, hence the weight diagram of any such bipartition has two crosses and two circles. Clearly the weight diagrams do not have any $\vee\wedge$ -pair, hence the corresponding modules are irreducible. \square

Lemma 6.2. *The composition factors of $S^i \otimes S^j$ which are not maximally atypical are given by the set*

$$R((i+j-k, k); (2^r, 1^{i+j-2r})), \quad k, r = 0, 1, \dots, \min(i, j), \quad k \neq r.$$

All these modules are $(n-2)$ -fold atypical irreducible.

Proof. This is again a recursive determination from the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ tensor products. As before the $S^i \otimes S^1$ and $S^i \otimes S^2$ -cases for $i \geq 1$ respectively $i \geq 2$ should be treated separately. For $S^i \otimes S^j$, $i, j \geq 3$ we obtain the regular formulas

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} &= (S^i + 2S^{i-1} + S^{i-2}) \otimes (S^j + 2S^{j-1} + S^{j-2}) \\ &= S^i \otimes S^j + \text{lower terms} \end{aligned}$$

where the lower terms are known by induction. In the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ tensor product the $R(\cdot)$'s from above cannot occur (for degree reasons) in any tensor product $\mathbb{A}_{S^p} \otimes \mathbb{A}_{S^q}$ for $p \leq i$, $q \leq j$ where either $p < i$ or $q < j$. Hence they cannot occur in any tensor product decomposition of any $S^p \otimes S^q$ for p, q as above, hence they have to occur in the $S^i \otimes S^j$ -decomposition. The number of these modules is $(\min(i, j)^2 - \min(i, j))$. Subtracting the inductively known numbers of not maximally atypical contributions in $S^p \otimes S^q$ in the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ -tensor product from the number of all such contributions in $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ we get $\min(i, j)^2 - \min(i, j)$ remaining modules. Hence there are no other summands in $S^i \otimes S^j$. \square

Lemma 6.3. *The irreducible representation $R((i+j-k, k); (2^r, 1^{i+j-2r}))$ is isomorphic to*

$$L(i+j-k, k, 0, \dots, 0 | 0, \dots, 0, -r, -i-j+r).$$

Proof. Let m denote the maximal coordinate of a cross or circle in the weight diagram of the bipartition. To obtain the weight diagram of the highest weight we have to switch all labels to the right of this coordinate as well as the first $M-n+2$ labels to its left which are not labelled \times or \circ by the explicit description of θ in [Hei14]. Since we have four such labels this amounts to switching all the labels at positions ≥ -1 and $< M$ (all of them \vee 's) and the $n-2$ \wedge 's at positions $-2, \dots, -n+1$ to \vee 's. The crosses are at the positions $i+j-k, k-1$ and the circles at the positions $i+j-r, r-1$. The result follows. \square

Lemma 6.4. *The lower atypical direct summands of $S^i \otimes S^j$ are given by the set*

$$R((i+j-k, k); (2^r, 1^{i+j-2r})), \quad k, r = 0, 1, \dots, \min(i, j), \quad k \neq r.$$

Proof. For any irreducible mixed tensors $R(\lambda), R(\mu)$ we have $\text{Ext}_1(R(\lambda), R(\mu)) = 0$ since every block contains a unique irreducible mixed tensor by [Hei14]. \square

For a maximally atypical weight $(\lambda_1, \dots, \lambda_n | -\lambda_n, \dots, -\lambda_1)$ denote by

$$L_0(\lambda_1, \dots, \lambda_n) \boxtimes L_0(-\lambda_n, \dots, -\lambda_1)$$

the underlying irreducible $Gl(n) \times Gl(n)$ -module. Denote by π the following additive map from irreducible $Gl(n) \times Gl(n)$ modules to irreducible $Gl(n|n)$ -modules:

$$\begin{aligned} \pi((L_0(\lambda_1, \dots, \lambda_n) \boxtimes L_0(\mu_1, \dots, \mu_n))) \\ = \begin{cases} 0 & L(\lambda_1, \dots, \lambda_n | \mu_1, \dots, \mu_n) \in \Gamma \\ L(\lambda_1, \dots, \lambda_n | \mu_1, \dots, \mu_n) & \text{else.} \end{cases} \end{aligned}$$

Corollary 6.5. *The not maximally atypical contributions to $S^i \otimes S^j$ are given by*

$$\pi((L_0(i, 0, \dots, 0) \boxtimes L_0(0, \dots, 0, -i)) \otimes (L_0(j, 0, \dots, 0) \boxtimes L_0(0, \dots, 0, -j))).$$

Corollary 6.6. *For $n = 2$ the tensor product $S^i \otimes S^j$ ($i > j$) decomposes as*

$$S^i \otimes S^j \simeq \begin{pmatrix} S^{i+j-1} \oplus Ber S^{i+j-3} \oplus \dots \oplus Ber^{j-1} S^{i-j+1} \\ S^{i+j}(1 + Ber^{-1}) + \dots + Ber^j S^{i-j}(1 + Ber^{-1}) \\ S^{i+j-1} \oplus Ber S^{i+j-3} \oplus \dots \oplus Ber^{j-1} S^{i-j+1} \end{pmatrix} \\ \oplus \pi((L_0(i, 0, \dots, 0) \boxtimes L_0(0, \dots, 0, -i)) \otimes (L_0(j, 0, \dots, 0) \boxtimes L_0(0, \dots, 0, -j))).$$

The tensor product $S^i \otimes S^i$ decomposes as

$$S^i \otimes S^i \simeq B^{i-1} \oplus \begin{pmatrix} S^{2i-1} \oplus Ber S^{2i-3} \oplus \dots \oplus Ber^{i-1} S^1 \\ S^{2i}(1 + Ber^{-1}) + \dots + Ber^i S^0(1 + Ber^{-1}) + B^{i-2} \\ S^{2i-1} \oplus Ber S^{2i-3} \oplus \dots \oplus Ber^{i-1} S^1 \end{pmatrix} \\ \oplus \pi((L_0(i, 0, \dots, 0) \boxtimes L_0(0, \dots, 0, -i)) \otimes (L_0(i, 0, \dots, 0) \boxtimes L_0(0, \dots, 0, -i))).$$

Remark. For $n = 2$ all the maximal atypical summands in $S^i \otimes S^j$ have nonvanishing superdimension. This is true for any $n \geq 2$ and follows from results in [Hei14], section 13.

Remark. One can show [HWng] that the projection of $S^i \otimes S^j$ ($i > j$) to the maximal atypical block is always indecomposable for $n \geq 2$ and is a direct sum of two indecomposable modules for $i = j$.

7. THE $Gl(3|3)$ -CASE AND A CONJECTURE

The method applied to compute the $S^i \otimes S^j$ tensor products in the $Gl(2|2)$ -case works in principal for arbitrary n . Note that the results on the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ tensor products are valid for any n . Furthermore we determined the part of $S^i \otimes S^j$ which is not maximal atypical for any $n \geq 2$, hence we restrict here to the maximal atypical part. The obstacle to use the method of the \mathcal{R}_2 -case effectively is that the composition factors of the modules $R(a, b)$ appearing in the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ -case are difficult to compute. Decomposing a few $R(a, b)$ for small a and b in the $n = 3$ -case and then computing the composition factors of the $S^i \otimes S^j$ tensor products recursively, we arrive at the following tensor products ($\Lambda^2 = (S^2)^\vee$).

$$S^1 \otimes S^1 \simeq \mathbf{1} \oplus \begin{pmatrix} S^1 \\ S^2 + \Lambda^2 + \mathbf{1} \\ S^1 \end{pmatrix} \\ . \\ S^2 \otimes S^1 \simeq \begin{pmatrix} S^2 \\ S^3 + [2, 1, 0] + S^1 + B^{-1} \\ S^2 \end{pmatrix} \\ S^3 \otimes S^1 \simeq \begin{pmatrix} S^3 \\ S^4 + [3, 1, 0] + S^2 \\ S^3 \end{pmatrix}.$$

$$\begin{aligned}
S^2 \otimes S^2 &\simeq [1, 1, 0] \\
&\oplus \left(\begin{array}{c} S^3 \oplus [2, 1, 0] \\ S^4 \oplus [3, 1, 0] \oplus [2, 2, 0] \oplus \Lambda^2 \oplus [2, -1, -1] \oplus [0, -1, -1] + S^2 + \mathbf{1} \\ S^3 \oplus [2, 1, 0] \end{array} \right). \\
S^3 \otimes S^2 &\simeq \left(\begin{array}{c} S^4 + [3, 1, 0] \\ S^5 + [4, 1, 0] + [3, 2, 0] + S^3 + [2, 1, 0] + [3, -1, -1] \\ S^4 + [3, 1, 0] \end{array} \right).
\end{aligned}$$

$$\begin{aligned}
S^3 \otimes S^3 &\simeq [2, 2, 0] \oplus \\
&\left(\begin{array}{c} S^5 + [4, 1, 0] + [3, 2, 0] \\ S^6 + [5, 1, 0] + [4, 2, 0] + B^{-1}S^5 + [3, 3, 0] + S^4 + [3, 1, 0] + [1, 1, 0] + [2, 2, 0] \\ S^5 + [4, 1, 0] + [3, 2, 0] \end{array} \right).
\end{aligned}$$

The conjectural general picture. For $n \geq 3$, $S^i \otimes S^j = M$ will be indecomposable if $i \neq j$. $S^i \otimes S^i$ splits as

$$[i-1, i-1, \dots, i-1, 0] \oplus M.$$

The socle of M is for $i \geq j$

$$\text{soc}(M) = [i+j-1, 0, \dots, 0] + [i+j-2, 0, \dots, 0] + \dots + [i, j-1, \dots, 0].$$

M has Loewy length 3. Note that since $\mathbb{A}_{S^i} \rightarrow \mathbb{A}_{S^j} \twoheadrightarrow S^i \otimes S^j$ and the maximal Loewy length of an $R(a, b)$ is 5, the Loewy length of M is at most 5.

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